REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (Leave blank)

2. REPORT DATE October 1, 1990

3. REPORT TYPE AND DATES COVERED Final Report, 15 Aug 87 to 29 Sep 90

4. TITLE AND SUBTITLE
NONLINEAR STABILITY IN FLUIDS AND PLASMA DYNAMICS-
CONFORMAL QUASI-CONFORMAL GEOMETRY: ITERATION, DISTORATION,
DYNAMICS AND CIRCLE PACKING

5. FUNDING NUMBERS
AF49620-87-C-0117
61101E 6241/00

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8. PERFORMING ORGANIZATION REPORT NUMBER
AFOSR-TR 90 1157

SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
AFOSR/NM
Bldg 410
Bolling AFB DC 20332-6448

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
F49620-87-C-0117

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release;
distribution unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

Work by Michael H. Freedman and collaborators, "A note on topology and magnetic energy in incompressible perfectly conducting fluids" by M. H. Freedman Journal of Fluid Mechanics 194:549-551 (1988) This note shows that in the low compressibility-high conductivity regime of magneto hydrodynamics and non-trivial linking between circular packets of H-integral curves implies a low bound to magneto energy (E).

14. SUBJECT TERMS

15. NUMBER OF PAGES

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR

October 1, 1990

Air Force Office of Scientific Research
Bolling Air Force Base
Washington, D. C. 20332-6448
REFERENCE: Contract F49620-87-C-0117

FINAL REPORT

Michael Freedman, Burt Rodin, their collaborators and graduate students have completed the proposed research under this contract. A summary of the results follows.

a. Work by Michael H. Freedman and collaborators

During this contract period research was done on dynamical questions with the focus on the propagation of distortion under groups of transformation or flows.

"A note on topology and magnetic energy in incompressible perfectly conducting fluids" by M. H. Freedman
Journal of Fluid Mechanics 194:549-551 (1988)

This note shows that in the low compressibility-high conductivity regime of magneto hydrodynamics *any* non-trivial linking between circular packets of **H**-integral curves implies a low bound to magnetic energy (E).

"Links of tori and the energy of incompressible flows"
by M. H. Freedman and Z.-X. He
Submitted to *TOPOLOGY*

The notion of conformal modulus of an annulus is borrowed from complex analysis and extended to solid tori in 3-space. It is shown that if such tori are linked in any topologically nontrivial fashion the modulus of at least one solid torus is less than the constant (125/48)p. This estimate is used to derive a lower bound to the kinetic energy of an incompressible fluid whose flow includes specified rotations about topologically linked tori. Previous work of V. I. Arnold ["The asymptotic Hopf invariant and its applications," Sel. Math. Sov. 5:327-345, 1986, English translation] implies a lower

bound in the special case that the linking is detectable by the abelian invariant: linking number.

"Factoring the logarithmic spiral" by M. H. Freedman and Z.-X. He

Inventiones mathematicae 92:129-138 (1988)

Distortion of length and angle are widely studied in geometry and dynamics. Much of the success in the study of one-dimensional complex dynamics flow from the Ahlfors-Bers-Sullivan theory [Ahlfors, L.V. & Bers, L. "Reimann's mapping theorem for variable metrics," Ann. Math. 72:384-404 (1960); and Sullivan, D. "Quasiconformal homeomorphisms and dynamics. I. Solution to the Fatou-Julia problem on wandering domains," Ann. Math. 122:401-418 (1985).] for deforming conformal structures -- essentially from the existence and uniqueness theory for the Beltrami equation $\partial_{\bar{z}} f / \partial_{z} f = m$.

No comparable analytic theory exists for the distortion of length in the plane or of length or angle in higher dimensions. (In these cases the local distortion is specified by more than d parameters in dimension = d, so the analogous equations are over determined.) In this paper we find a constraint on a method of returning a quasi-isometry to the identity: factorization into maps of small (isometric) distortion. We study the following L-quasi-isometry of the closed unit disk $\bar{D} = \{(x_1, x_2);\ x_1^2 + x_2^2 \leqq 1\}$:

$$S_k(r,q) = (r, q + k \log r), \quad k = L - \frac{1}{L},$$

where $L > 1$, (r,q) are the polar coordinates of $\mathbf{R}^2$ and log denotes natural logarithm. We find that it requires $N \leqq k/$÷à2-1 factors to write S_k is L^2-quasiconformal, the minimal number of factors of S_k with conformal distortion $\leqq c$ grows like $2 \log_c k$ when k goes to infinity. Roughly, angular distortion may be removed in logarithmic time but distortion of length cannot be removed faster than linearly.

"A remark on inherent differentiability" by M. H. Freedman and Z.-X. He

Proceedings of the American Mathematical Society 104:1305-1310 (1988)

Harrison's analysis of C^r-diffeomorphisms which are not conjugate to C^s-diffeomorphism for $s > r > 0$ is extended to dimension = 4. Also topological conjugacy may be generalized to an arbitrary change of differentiable structure. Combining these statements yields: for any smooth manifold of dimension ≥ 2 there is a C^r-diffeomorphism which is not a C^s-diffeomorphism w.r.t. any smooth structure.

"A power law for the distortion of planar sets" by M. H. Freedman

Discrete and Computational Geometry 2:345-351 (1987)

We consider how to map the sites of a square region of planar lattice into a three-dimensional cube, so as to minimize the maximum distortion of distance. We consider the cube to be endowed with a "foliated" geometry in which horizontal distance is standard but vertical communication only occurs at the surface of the cube. These geometries may naturally arise of a planar data set is to be stored in a stack of chips. It is proved that any one-to-one map which fills the cube with a fixed "density" must produce a distortion of distance which grows as the one-sixth power of the diameter of the square and the two-thirds power of the density. Moreover, we explicitly define one-to-one maps with 100% density, one-sixth power stretching, and a small leading coefficients.

"Strange actions of groups on spheres, I and II"
by M. H. Freedman and Richard Skora
(I) *Journal of Differential Geometry* 25:75-98 (1987)
(II) Holomorphic Functions and Moduli, David Drasin, Ed., Vol. II, 41-57, Springer-Verlag (1988)

In these papers we investigated certain topological analogs of Schottky groups, called admissible actions, and their compatibility with various structures on spheres. We constructed an action $f : F^2$ ¥ S^3 Æ S^3 which was not topologically conjugate to a uniformly quasiconformal action. Also, there is an example $y : (F^r$ ¥ $\mathbf{Z}_2)$ ¥

S^3 Æ S^3 , r sufficiently large, which is smooth and uniformly quasiconformal, but not topologically conjugate to a conformal action. We gave examples of admissible actions on higher dimensional spheres, and analyze the structures preserved.

"Solving Beltrami equations by circle packing" by Z.-X. He
To appear in *Transactions of American Mathematical Society*

Andreev's Theorem on the existence of circle packings to construct approximating solutions to the Beltrami equations on Riemann surfaces. The convergence of the approximating solutions on compact subsets will be shown. This gives a constructive proof of the existence theorem for Beltrami equations.

"An estimate for hexagonal circle packings" by Z.-X. He
Submitted to the *Journal of Differential Geometry*

In B. Rodin and D. Sullivan's paper ["The convergence of circle packings to the Riemann mapping," J. Differential Geometry 26:349-360, 1987] they showed that any circle packing which is combinatorially equivalent to an infinite regular hexagonal circle packing should also be regular hexagonal, and as a consequence, packing stability constant s_n converges to 0. They conjectured that $s_n \leq C/n$ for some constant C . This paper proves the conjecture. The estimate for s_n is best possible
$s_n \geq 4/n$.

b. Work by B. Rodin and collaborators

During the contract period work was done on circle packings, conformal mappings, and numerical conformal geometry.

"The convergence of circle packings to the Riemann mapping"
by B. Rodin and D. Sullivan.
J. Differential Geometry, **26** (1987), 349-360.

This paper gives the proof of Thurston's conjecture that the Riemann mapping function from a bounded simply connected

plane region to the unit disk is the limit of circle packing isomorphisms.
A region R is packed with small circles of radius e. By means of Thurston's algorithm an isomorphic packing of the unit disk is created; see Figure 1. Thurston conjectured that this correspondence of circles converges to the Riemann mapping function as e Æ 0. The proof consists of three steps. First, the Ring Lemma is proved; this gives an a prioi theoretical bound on the dilatation of the circle correspondence (explicit estimates were later given by L. Hanson [On the Rodin and Sullivan ring lemma, Complex Variables, 10 (1988), 23-30]) and allows one to conclude that the family is equicontinuous as e Æ 0. The Hexagonal Packing Lemma shows if two tangent circles are at the center of n generations of a circle packing with the combinatorics of the hexagonal circle packing pattern then the ratio of the radii of the circles must Æ1 as n Æ ∞ . This shows that the limit of the q.c.-mappings is conformal or else constant. A Length-Area Lemma is then proved; it shows that the limit mapping into the unit disk is surjective.

"Schwarz's lemma for circle packings" by B. Rodin.
Invent. Math., **89** (1987), 271-289.

The familiar hexagonal circle packing by circles of constant radius is denoted HCP(n), where n ≤ ∞ is the number of generations in the pattern. Circle packing with the combinatorics of HCP(n) but with circles of not necessarily constant radius are denoted by HCP'(n). The Hexagonal Packing Lemma of Rodin-Sullivan [loc. cit.] states that a circle packing of type HCP'(∞) is actually of type HCP(∞). The main result of this paper is the Schwarz lemma type result: Let P be an HCP(n) packing and let D be the smallest disk which contains P. Let P' be and HCP'(n) packing which is contained in D. Then the radii r,r' of the generation zero circles of P and P' satisfy r' ≤ ar where a is a universal constant.

The proof of this result involves three different techniques. The first is to establish a discrete potential theory on the lattice HL(n) consisting of n generations about the origin of the standard hexagonal (or triangular) lattice {m + ne$^{ip/3}$: m,n Œ Z}. The radii of the circles of a packing P' of type HCP'(n) determine a discrete positive function rad P' defined on HL(n).

This function is subharmonic in the discrete sense, that is, its value at an interior point of HL(n) is no greater than the average of its value at the six neighboring points. In order to do discrete potential theory in this context we need an analog of the logarithm function. We prove: There is a unique function l:HL(∞) Æ R which is discrete harmonic except at 0 and which satisfies l(a) - log|a| = O(1/|a|) as a Æ ∞ in HL(∞).

The second technique is needed for the passage from discrete to continuous potential theory. This is needed to prove the lemma: Let u be a discrete subharmonic function defined on HL(n). Then the
u(0) ≤ cS where S is the average of the values of u on the 6n boundary points of HL(n) and c is an absolute constant. The continuous analog of this lemma can be proved by estimating the Poisson kernel for a hexagonal region; our proof of the discrete case makes use of this continuous analog and of estimates from finite difference theory of the error between continuous and discrete solutions of Dirichlet problems.

The third technique is to use the length-area method to obtain the Schwarz lemma analog from the above lemma on discrete positive subharmonic functions.

"An extremal region for harmonic measure"
by B. Rodin and S. E. Warschawski.
Complex Variables, **9** (1987), 271-282.

Let W be a simply connected region containing the fixed point
r > 1. Let b_W be the portion of the boundary of W which is contained in the closed unit disk **D**. Let G_W be the family of crosscut chains in W which separate b_W from r. In Theorem 1 we find the region which makes the value of the modulus (reciprocal extremal length) of G_W a minimum. This result can be considered as an extremal region problem in the sense of Grotzsch, Mori, and Teichmuller extremal problems. When we calculate the modulus for our extremal region we obtain as a corollary the following harmonic measure estimate due to A. Ostrowski: $w(r,b_W,W) \le (2/p)\ \arcsin(2\dot{+}r/(r+1))$. Ostrowski used a different definition of harmonic measure and his proof is

extremely complicated. We show that our definition of harmonic measure by Perron's method implies Ostrowski's result.

"Schwarz's lemma for circle packings, II" by B. Rodin.
J. Differential Geometry, **30** (1989) 539-554.

This paper contains a number of results on the derivative conjecture and on explicit error bounds for the circle packing approximation to the Riemann mapping function. The derivative conjecture states that in the circle packing approximation, the ratio of the radii of an image circle to its preimage circle converges to the modulus of the derivative of the mapping function, uniformly on compacta. It follows from the results of Rodin and Sullivan that the convergence does hold in the L^p norm on compacta $(p > 2)$. In this paper we show that it holds in the BMO norm on compacta. We also so that certain growth rates on the hexagonal packing constants s_n introduced in Rodin-Sullivan [loc. cit.] imply the validity of the derivative conjecture. The strongest theorem of this kind is: If $s_n = o(1/\log^2 n)$ then the derivative conjecture is true. (Subsequently, Z.-X. He [he] proved that $s_n = O(1/n)$ and hence the the derivative conjecture is true.)

Consider the error term $|f_e(z) - f(z)|$ where f is the Riemann mapping function and f_e is the circle packing approximation using circles of radius e. We show that this term is of order at most
$\div s_{[1/2e^{1/2}]} + e^{1/4}$ on compacta. When we combine this with He's result [he] we see that the error term is at most of order $e^{1/4}$ on compacta.

"On Thurston's proof of Andreev's theorem"
by B. Rodin and A. Marden. To appear in *Computational methods and function theory,* Springer lecture notes.

In his Princeton notes, Thurston states a general result regarding the existence and uniqueness of circle packings of prescribed combinatorial type on closed surfaces. His theorem treats separately the cases of genus $g = 1$ and $g \geq 2$. He states that the case $g = 0$, which is not proved in these notes, is equivalent to a theorem of E. M. Andreev [Mat. Sb., Nor. Ser. 83 (1970) 256-260]. It is implicit in Thurston's notes that the

continuity method used there to prove the higher genus case might be modified to give a proof of the genus 0 case as well. Such a proof would be very different from that of Andreev.

We succeed in modifying Thurston's argument so that it yields a proof in the genus 0 case. We also include Thurston's extension which allows one to prescribe not only the pattern of the packing but the intersection angles as well as tangencies.

"An inverse problem for circle packing approximations to the Riemann mapping function" by I. Carter and B. Rodin. To appear in the *Transactions of the Americam Mathematical Society.*

A number of results are obtained on the discrete potential theory of the hexagonal grid. These results are used to prove the convergence of the circle packing isomorphism from the unit disk to the region under consideration. Whereas Kleinian group theory was the main tool for proving the convergence of the isomorphism from the region to the circle, discrete potential theory and finite difference theory are the main tools needed for the inverse problem. This result provides a conformal triangular grid for a plane region (the nerve of the circle packing which corresponds to a regular circle packing in the disk) which can be used for solving elliptic PDE in the region by finite difference methods.

"Circle packing and conformal mapping" by I. Carter. Ph. D. Dissertation, University of California, San Diego, 1989.

The circle packing approximation to the Riemann mapping function is shown to be valid for mutiply connected regions. In this case the image region is a Koebe circle domain, that is, a disk with disks removed.

The circle packing algorithm of Thurston is proved to converge. It is proved that any triangulated bordered surface of genus 0 or 1 has a unique flat radius function defined on the vertices with prescribed boundary values. In case the surface is a simply connected plane region, it is proved that the corresponding circle packings converge to a conformal immersion with prescribed derivative modulus.

Numerical experiments are given which estimate the error in the circle packing approximations to the Riemann mappings function. An idea of Thurston for reducing the error due to boundary irregularity is tested numerically and found to be very effective.

"Circle packing and Riemann surfaces" by David Minda and Burt Rodin, submitted for publication.

The authors consider triangulated compact bordered surfaces. They show that there is an unique Riemann surface structure which can be put on such a surface so that the triangulation is the nerve of a circle packing on that Riemann surface. This result is then applied to yield a constructive method for estimating the hyperbolic metric. One approximates the plane region with the triangular grid of mesh e and obtains a circle packing and hyperbolic metric on the corresponding Riemann surface. It is shown that these metrics converge to the actual hyperbolic metric as e tends to 0.

"On a problem of A. Beardon and K. Stephenson" by Burt Rodin. To appear in the *Indiana University Mathematics Journal.*

In their paper "Circle packings and uniformization", Beardon and Stephenson pose the question of uniqueness for disk realizations of an infinite circle packing nerve of hyperbolic type. The present paper solves this problem by showing that the disk realization is unique up to linear fractional transformations.

"Canonical conformal mapping for a multiply connected domain by Thurston's circle packing method".

Computer software created by I. Carter.

Version I for UNIX, X-windows environment.

Version II for Apple Macintosh II or IIx personal computer.

Version III for Ardent Titon.

We describe the version II package. A window appears and the user is asked to draw a region of finite connectivity with the mouse. When he clicks on "done" he is asked what size circle to use to fill the region. He responds by adjusting a size scale with the mouse. He then clicks on "fill" and the region is filled with circles of the proper size. The user is then asked to click on the circle which is to be sent to the origin. When the user does this, the computer then changes the radii of the packed circles so that they become packed

inside a Koebe circle domain rather than the region. The result is a circle packing approximation to the canonical circle domain conformal mapping.